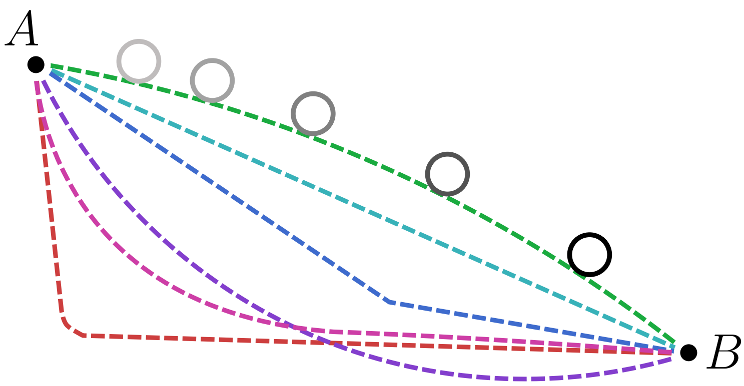
Introduction

Perhaps one of the most well-known problems in the history of mathematics is the one posed by Johann Bernoulli in 1696, who addressed the readers of the scientific journal *Acta Eruditorum*:

> “ *I, Johann Bernoulli, address the most brilliant mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem, whose possible solution will bestow fame and remain as a lasting monument . . . If someone communicates to me the solution of the proposed problem, I shall publicly declare him worthy of praise.* ”

The problem was simple—to find the **quickest** path from two vertically coplanar points A and B for an object to travel that is affected only by gravity:



The solution, however, is much more difficult to derive, and it might seem counter-intuitive at first. When Bernoulli first posed this problem to the mathematical figures of his time, he originally provided 6 months of time in which expected solutions to be given. However, none were received, and at the request of Leibniz (another notable mathematician who is often considered the co-founder of calculus), Bernoulli extended the period by another one and a half years.

By the end of the two-year period in 1698, five mathematicians had submitted their solutions: Jakob Bernoulli (Johann's brother), Issac Newton, Gottfried Leibniz, Ehrenfried Walther von Tschirnhaus, and Guillaume de l'Hôpital. Each with their own merits, one of the most interesting and approachable solutions was developed by Johann himself, and deals with an intuitive understanding of physics and the behavior of light.

Bernoulli's Solution

For the motion of an ideal particle where the net force is the resultant of only weight and the normal force, the principle of the conservation of energy can be used to derive an equation for the final velocity of a particle, v_f, given the magnitude of its vertical displacement from rest, y (Equation 1):

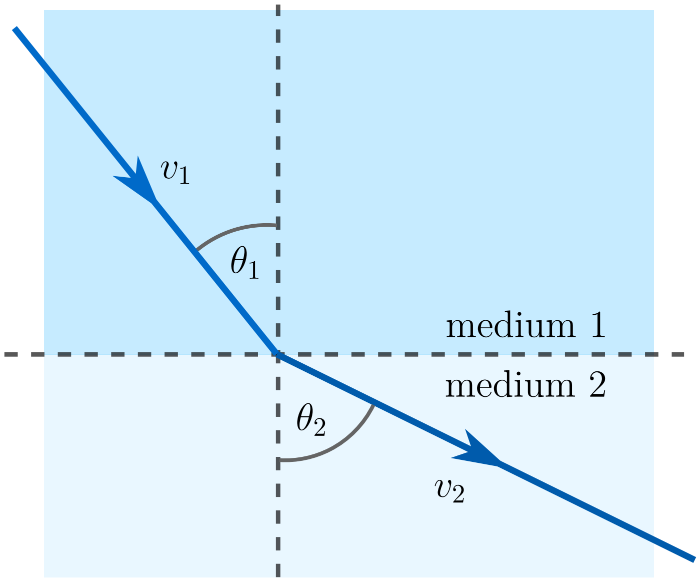
$$\begin{aligned} \Delta E_{mech} &= 0 \\ E_{mech,i} &= E_{mech,f} \\ \cancelto{0}{K_i} + U_{g,i} &= K_f + \cancelto{0}{U_{g,f}} \\ \cancel{m}gy &= \frac{1}{2}\cancel{m}v_f^2 \\ v_f &= \sqrt{2gy} \qquad (1) \end{aligned}$$

Because this equation, and the conservation of energy model as a whole, is path-independent, it can be used to determine the speed of the object at any height on the generalized Brachistochrone curve AB.

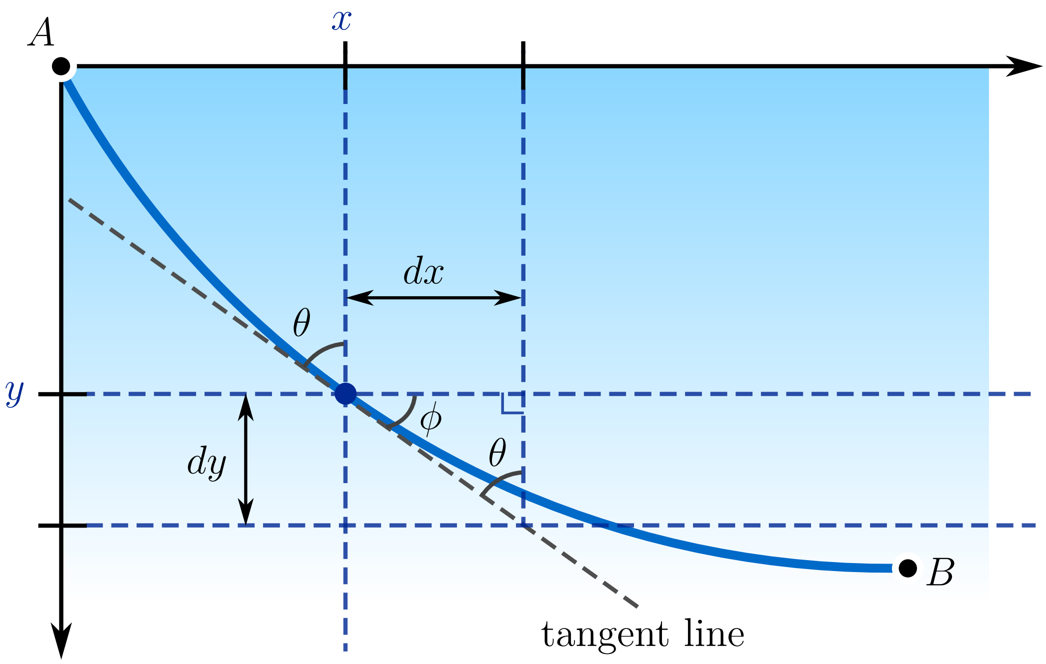
Another key to his solution was his consideration of **Snell's Law** (Equation 2), which describes how light is refracted as it travels across a boundary between media of varying density:

$$\begin{aligned} \frac{\sin\theta_1}{\sin\theta_2} &= \frac{v_1}{v_2} \\ \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} &= k \qquad (2) \end{aligned}$$

where θ_1 and θ_2 are the angles of incidence and refraction (the angles between the initial and resulting rays of light and the normal of the boundary of the surface), v_1 and v_2 are the initial and resulting phase velocities, and k is some constant:

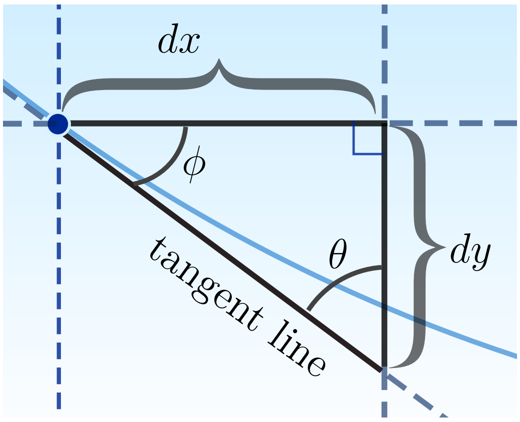


Using these models, Bernoulli solved the problem with an elegant thought experiment. He considered some non-uniform optical medium where the density becomes continuously less from top to bottom. If light was shone through, it would get refracted and become increasingly faster according to **Snell's Law**. Furthermore, by carefully controlling the density at various points, the velocity at those same points could be controlled such that they are the same as Equation 1. By doing this, the optical medium is now a model for the Brachistochrone problem, where A is a point at the top, and B is a point at the bottom:



According to **Fermat's Principle**, or the **principle of least time**, the path that light takes is always the one where light can travel along it in the shortest time possible. This makes the equation of this curve the **quickest** solution to the Brachistochrone problem.

Enlarged, the triangle made by the tangent line, dy, and dx appears as follows:



From this triangle, $\sin\theta$ can be manipulated as follows to create an expression in terms of dx and dy:

$$\begin{aligned}\sin\theta &= \cos\left(90^\circ - \theta\right) = \cos\phi \\ &= \frac{1}{\sec\phi} = \frac{1}{\sqrt{\sec^2\phi}} \\ &= \frac{1}{\sqrt{1+\tan^2\phi}} = \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\end{aligned}$$

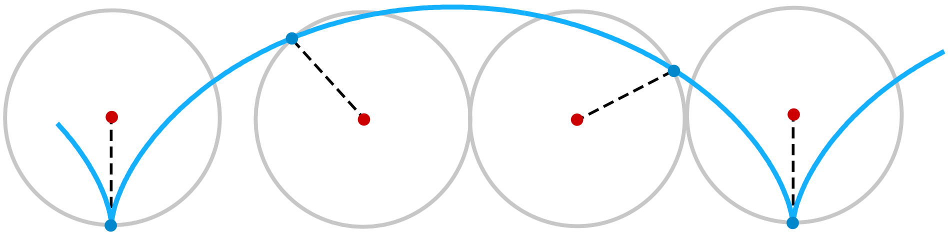
Substituting this into Equation 2 as $\sin\theta$ gives:

$$\frac{1}{v\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = k$$

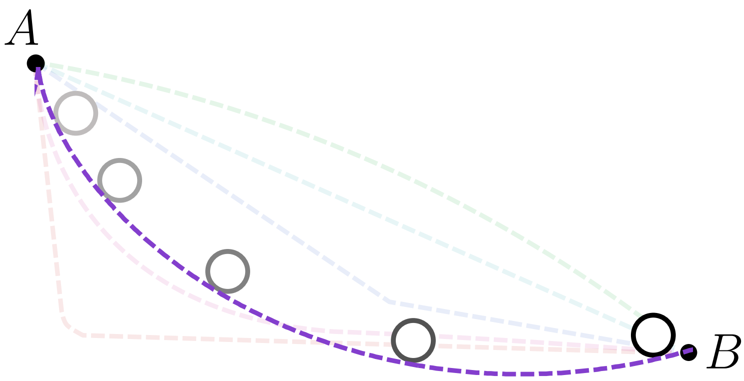
where k is still some constant. Finally, by substituting Equation 1 for v, the following equation is produced, which can be solved for $\frac{dy}{dx}$ to produce the differential equation labeled Equation 3.

$$\begin{aligned}\frac{1}{\sqrt{2gy}\sqrt{1+\left(\frac{dy}{dx}\right)^2}} &= k \\ y\left[1+\left(\frac{dy}{dx}\right)^2\right] &= \frac{1}{2gk^2} \rightarrow C \\ 1+\left(\frac{dy}{dx}\right)^2 &= \frac{C}{y} \\ \left(\frac{dy}{dx}\right)^2 &= \frac{C}{y} + 1 \qquad (3)\end{aligned}$$

This new equation is immediately apparent to be the differential equation for a cycloid, a fascinating curve generated by a point on the end of a rolling circle:



Through this, Bernoulli showed that the **quickest** solution to his own Brachistochrone problem is indeed an inverted cycloid, which is the path an object will take to get from A to B in the shortest time possible:



Computational Model

Bernoulli's result can certainly be confirmed by using more rigorous proofs involving tools from the calculus of variations and optimal control, but those derivations are far beyond the scope of this project. Instead, a much easier method was used to provide a second confirmation of the solution—a computational model.

Overview

Programmed in the language Java, the simulation program models real-world physics through a mock-continuous approach. While true continuous simulation is impossible without implementing multiple complex calculus-based models, a trick *was* borrowed from the fundamentals of calculus: that the rate of change of a quantity, such as position in the case of velocity, is based off of its change given a small, finite value of time dt as $dt \to 0$. In other words:

$$v := \frac{dx}{dt} = \lim_{dt \to 0} \frac{\Delta x}{dt}$$

Figure 1: Originally developed by Sun Microsystems and currently owned by Oracle, Java and its related software currently run on over three billion devices worldwide

Instead of modeling the velocity of the sliding masses in the Brachistochrone simulation as an instantaneous rate of change, it was modeled as the rate of change over extremely small values of dt; in this case, the default is $1/120,000^{\text{th}}$ of a second, or $0.000008\overline{333}$ seconds. By doing this, the values given from the simulation become extremely close approximations for the real-world results, and as such, provide another way to validate Bernoulli's solution.

The source code is hosted on a repository from **github.com**, an open-source development version control service based off of **git**. A link is available through the QR code to the left.

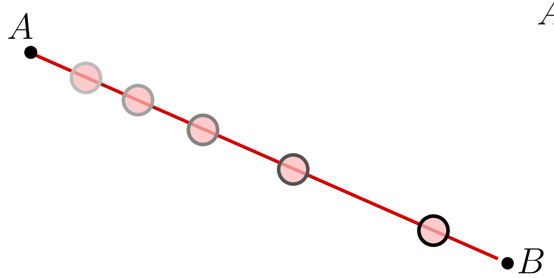
Approach

While each technical detail won't be discussed at length, one of the key aspects of the computational model is its use of *abstraction*, or the concept that if something can be generalized, it should. With this principle in mind, three different implementations of the `Curve` class were developed:

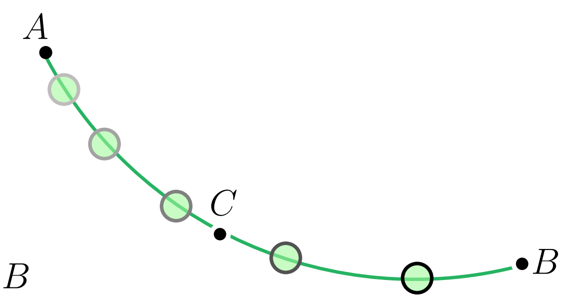
1. `StraightTrack`, which represents the straight path that travels directly from A to B (the shortest distance possible)

2. `CircularArc`, which models a segment of the circle that passes through three points: A, B, and some other controlling point C

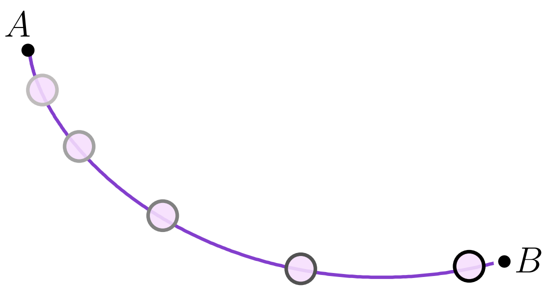
3. `CycloidArc`, which represents the cycloidal arc that intersects points A and B



(a) `StraightTrack`



(b) `CircularArc`



(c) `CycloidArc`

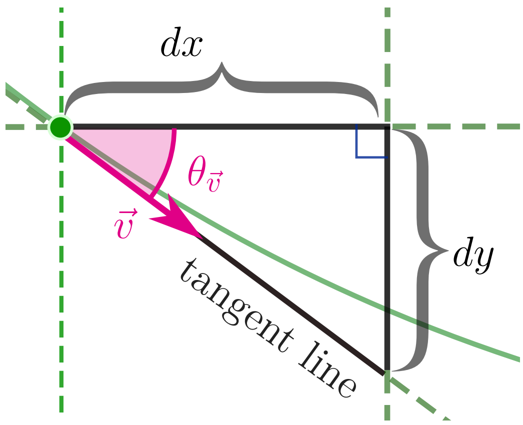
Creating a simulation for the different path shapes was, for the most part, relatively simple, as for some width and height, curves (a) and (c) are constrained to only one possibility. (in this case $w = 5$ m $\;h = 3$ m).

However, for the circular arc, (b), there exist an infinite number of paths that connect point A to B. In order to determine the *quickest* one with which to compare to others, an optimization with respect to time had to be performed (See 'Circular Arc Optimization')

For modeling the masses' motion over the different paths, Equation 1 was used, which describes how the magnitude of the velocity vector, $|\vec{v}|$, at any given point along a path is proportional to the square root of its vertical position:

$$|\vec{v}| = \sqrt{2gy}$$

where y is the vertical distance that the object has traveled since it was at rest. Then, $\vec{v}$'s angle, $\theta_{\vec{v}}$, can be calculated by taking the arctan of the derivative, $\frac{dy}{dx}$, at any given point, as the velocity vector is always colinear to the tangent line:



$$\tan\theta_{\vec{v}} = \frac{dy}{dx}$$
$$\theta_{\vec{v}} = \arctan\left(\frac{dy}{dx}\right)$$

Given $|\vec{v}|$ and $\theta_{\vec{v}}$, the horizontal and vertical components of the velocity can be determined:

$$v_x = |\vec{v}|\cos\theta_{\vec{v}}$$
$$v_y = |\vec{v}|\sin\theta_{\vec{v}}$$

Finally, these can be *iteratively* applied to the position of the mass, $\vec{r}$ for each dt by adding dx and dy to r_x and r_y, respectively:

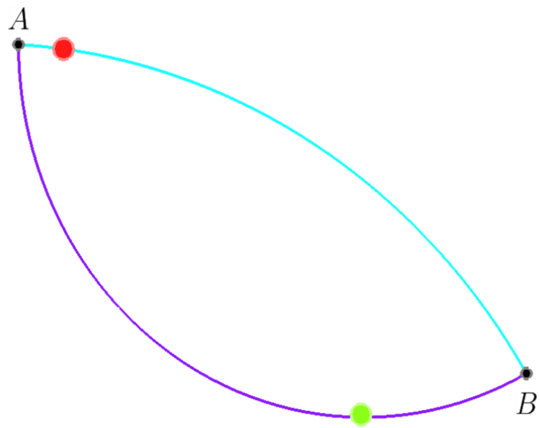
$$v_x := \frac{dx}{dt} \qquad v_y := \frac{dy}{dt}$$
$$dx = dt|\vec{v}|\cos\theta_{\vec{v}} \qquad dy = dt|\vec{v}|\sin\theta_{\vec{v}}$$

Circular Arc Optimization

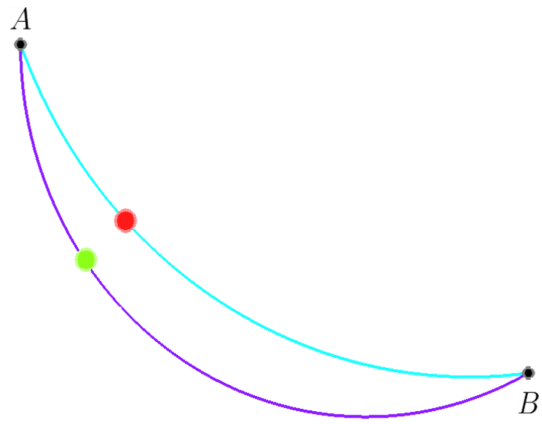
In order to find the *quickest* circular path possible from A to B, a common optimization algorithm called **gradient descent** was used, which allows the time from A to B, t_{AB}, to be minimized. The process is simple:

1. First, the two extremes are tested in order to find an estimate for the gradient. In this case, the two circular arcs in picture (a) are simulated, and, depending on which is lower on the *time gradient*, that is, which gets the mass from A to B in a lesser time, the algorithm moves closer to that side.

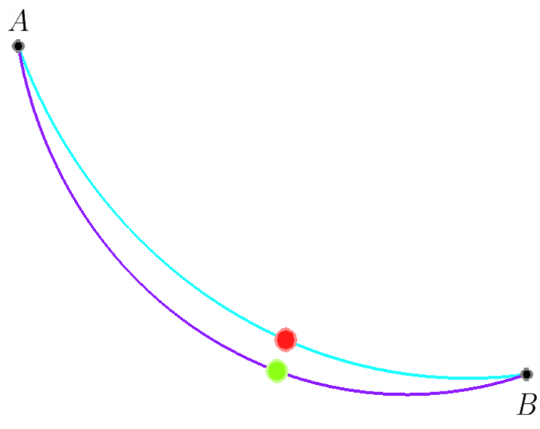
2. Next, the process repeats to a depth n, slowly converging to a *local minima* of the arc-space, which, in this case, is also the absolute minimum. This convergence is displayed in images (b-d)



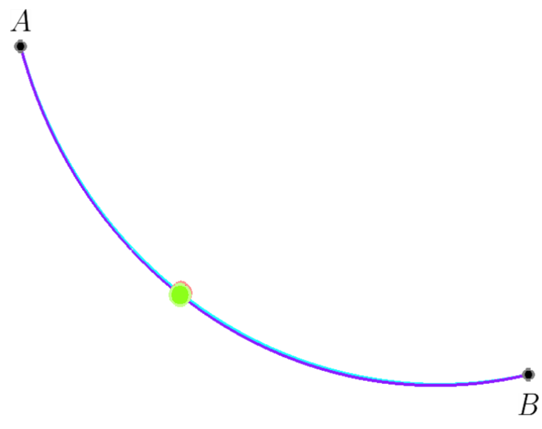
(a) $n = 1$: The initial iteration, showing the two extreme values for steepness at the top and bottom



(b) $n = 3$: Iteration 3 after the algorithm stepped down the gradient twice by becoming more steep



(c) $n = 4$: Iteration 4 when the algorithm decided decreasing the steepness would minimize time



(d) $n = 8$: Iteration 8, where the two simulations begin to converge to the optimal and fastest circular solution

Results

Using the three curves outlined in 'Approach,' each was tested to determine their relative speed in a simultaneous simulation as shown to the right. With a Δx of 5 meters and a Δy of -3 meters, the following results were derived for the time it takes for the mass to get from A to B, t_{AB}, in seconds: (Note that for the final results, an extremely small dt was used in order to attain a high level of approximation: for this, $dt = 1/960,000$ sec or $1.041\overline{666} \times 10^{-6}$ sec)

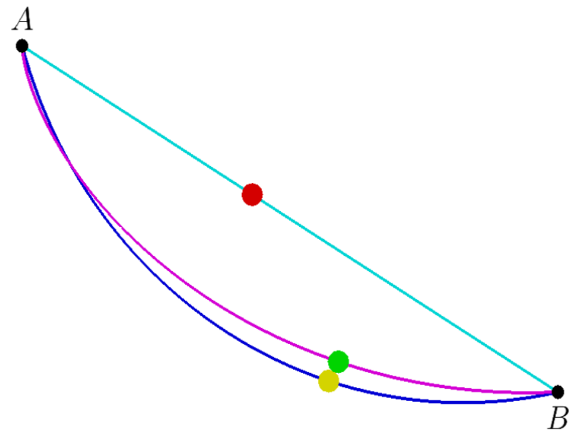


Figure 5: The final simulation screen, showing the three different `Curves`: `StraightTrack` (turquoise), `CircularArc` (violet), and `CycloidArc` (magenta)

`Curve`	t_{AB} (sec)
`StraightTrack`	1.5116
`CircularArc`	1.2679
`CycloidArc`	1.2616

Using basic physics calculations, the time it takes for an object to travel from the start of a 5×3 meter frictionless ramp can be found to be approximately 1.52031 seconds, making the predictions derived via simulation supposedly within only a <u>0.57%</u> error.

And, as is clearly shown by the results, the cycloid track is definitively the *quickest* solution to the Brachistochrone problem, beating the straight track and even the quickest possible circular arc track (albeit by very little). Furthermore, the results show that the *shortest* solution, the straight track, is definitely not the *quickest* one.